Facets of Tunneling: Particle production in external fields

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This paper presents a critical review of particle production in an uniform electric field and Schwarzschild-like spacetimes. Both problems can be reduced to solving an effective one-dimensional Schrodinger equation with a potential barrier. In the electric field case, the potential is that of an inverted oscillator $-x^2$ while in the case of Schwarchild-like spacetimes, the potential is of the form $-1/x^2$ near the horizon. The transmission and reflection coefficients can easily be obtained for both potentials. To describe particle production, these coefficients have to be suitably interpreted. In the case of the electric field, the standard Bogoliubov coefficients can be identified and the standard gauge invariant result is recovered. However, for Schwarzchild-like spacetimes, such a tunnelling interpretation appears to be invalid. The Bogoliubov coefficients cannot be determined by using an identification process similar to that invoked in the case of the electric field. The reason for such a discrepancy appears to be that, in the tunnelling method, the effective potential near the horizon is singular and *symmetric*. We also provide a new and simple semi-classical method of obtaining Hawking's result in the (t, r) co-ordinate system of the usual standard Schwarzchild metric. We give a prescription whereby the singularity at the horizon can be regularised with Hawking's result being recovered. This regularisation prescription contains a fundamental asymmetry that renders both sides of the horizon dissimilar. Finally, we attempt to interpret particle production by the electric field as a tunnelling process between the two sectors of the Rindler metric.

I. INTRODUCTION AND SUMMARY

In this paper, we present a critical review of particle production in Schwarzchild-like spacetimes and in an uniform electric field in Minkowski spacetime. Our emphasis here is on the tunnelling picture used to interpret results obtained by other (presumably more reliable!) methods.

Consider the problem of a scalar field propagating in flat spacetime in an uniform electric field background. The total particle production due to the presence of the electric field upto the one-loop approximation is correctly calculated by the gauge invariant method proposed by Schwinger [1]. The same problem can be reduced, in a time dependent gauge, to an equivalent Schrodinger equation with an inverted harmonic oscillator potential. Such an equation can be solved exactly using the standard flat spacetime quantum field theoretic methods. Since the problem is explicitly time dependent, the vacuum state at $t \to -\infty$ and at $t \to \infty$ are not the same. The Bogoliubov coefficients between the "in" and "out" vacua are easily calculated and the total particle production turns out to be the same as caluclated by the Schwinger method. However, if a space dependent gauge is used to describe the same field, the vacuum state of the field remains the same for all time and hence no particle production can take place. To recover the standard result, the tunnelling interpretation is introduced. This interpretation is useful since it provides a dynamical picture of particle production (see, for example [2]) and is the only way in which the standard gauge invariant result can be recovered in a time independent gauge. In this paper, we attempt a tunnelling description for both time dependent and time independent gauges. The method of complex paths, enunciated by Landau in [10], is used to calculate the transmission and reflection coefficients for the equivalent quantum mechanical problem. Then, an interpretation of these coefficients, in order to obtain the standard gauge invariant result is provided.

Let us next consider Schwarzchild-like spacetimes like the usual black hole spacetime, the Rindler spactime and the de-Sitter spacetime. In the standard black-hole spacetime, particle production was obtained by Hartle and Hawking [4] using semi-classical analysis. In this method, the semi-classical propagator for a scalar field propagating in the Schwarzchild spacetime is analytically continued in the time variable t to complex values. This continuation gives the result that the probability of emission of particles from the past horizon is not the same as the probability of absorption into the future horizon. The ratio between these probabilities is of the form

$$P[\text{emission}] = P[\text{absorption}]e^{-\beta E} \qquad (1)$$

where E is the energy of the particles and $\beta = 1/8\pi M$ is the standard Hawking temperature. The above relation is interpreted to be equivalent to a thermal distribution of particles in analogy with that observed in an atom interacting with black body radiation. In the latter case, the probability of emission of radiation by the atom is related to the probability of absorption by the atom by a similar relation as given above. Now, in Hawking's derivation, the Kruskal extension is of vital importance in obtaining the thermal spectrum. In this paper, we propose an alternate derivation of Hawking radiation without using the Kruskal extension. We consider the semi-classical propagator in the standard Schwarzchild co-ordinates near the horizon and then use a prescription to bypass the singularity there. This prescription gives the same result as that obtained by Hawking and can be used in all spacetimes with a Schwarzchild-like metric. We then attempt to create a tunnelling picture on the lines of that done for the electric field. The equivalent Schrodinger potential in this case is of the form $-1/x^2$ near the horizon. The transmission and reflection coefficients can easily be obtained using Landau's method of complex paths. To obtain particle production, these coefficients have to be suitably interpreted. In the case of the electric field, the standard Bogoliubov coefficients can be identified by recasting the normalisation rule between the transmission and reflection coefficients, namely $T + R = 1$, in such a way that the standard gauge invariant effective lagrangian result is recovered. However, in the case of the Schwarzchild-like spacetimes, such a tunnelling interpretation appears to be invalid. The Bogoliubov coefficients cannot be identified using the transmission and reflection coefficients in the same way as was done in the case of the electric field since the spectrum of created particles is then not thermal as Hawking's semiclassical method suggests. But reinterpreting these coefficients also does not give the standard result. The reason for such a discrepancy appears to be that in the tunnelling method, the effective potential near the horizon is symmetric and singular and hence both sides of the horizon appear to be the same.

Finally, we attempt to link particle production by an uniform electric field with processes occuring in the Rindler frame. We propose that tunnelling between the two Rindler sectors gives rise to the production of real particles. We do this entirely in a heuristic manner and show that this tunnelling process between the two Rindler sectors gives rise to the exponential factors in the expression for the effective lagrangian (see Eqn. (104). But the coefficients multiplying the exponential factors, however, must be determined by solving the problem exactly.

The layout of the paper is as follows. In section (II), we will discuss first the calculation of the reflection and transmission coefficients using Landau's powerful method of complex paths for the potentials $-x^2$ and $-1/x^2$. Then, in section (III) we apply the results of the previous section to particle production in an electric field. In section (IV), the semi-classical derivation of Hawking radiation without taking recourse to the Kruskal extension will be extensively discussed. Next, the reduction of the problem to an effective Schrodinger equation and tunnelling at the horizon will be considered. Finally, in section (V), we attempt to link particle production in the electric field to processes occuring in the Rindler frame.

II. FACETS OF TUNNELING

In this section we briefly review the basic concepts of semi-classical quantum mechanics in one dimension and formally describe the tunnelling process. We then apply the formalism to two potentials, namely, $V_1(x) = -x^2$ and $V_2(x) = -1/x^2$, and calculate the transmission and reflection coefficients for both.

A. Semi-classical limit of Quantum Mechanics

Consider a simple one dimensional quantum mechanical system with an arbitrary potential $V(x)$ where x denotes the space variable (see Ref. [10] for details). To describe the transition of the system from one state to another, we first solve the corresponding classical equations of motion and determine the path of transition. This path is, in general, complex since many processes like tunneling through a potential barrier cannot occur classically. Therefore, the transition point q_0 where the system formally makes the transition is a complex number determined by the classical conservation laws. Then, the action S for the transition from some initial state x_i to a final state x_f given by

$$S(x_f, x_i) = S(x_f, q_0) + S(q_0, x_i) \qquad (2)$$

is calculated. Here, $S(q_0, x_i)$ is the action for the system to move from the initial state x_i to the transition point q_0 while $S(x_f, q_0)$ is that to move from q_0 to x_f. The probability P for the transition to occur is given by the formula

$$P \sim \exp\left(-\frac{2}{\hbar}\text{Im}[S(x_f, q_0) + S(q_0, x_i)]\right). \qquad (3)$$

The above formula is valid only when the exponent of the exponential is large. Further, if the potential energy has singular points, these must also be considered as possible values for q_0. If the position of the transition point is not unique, then it must be chosen so that the exponent in Eqn. (3) has the smallest absolute value but still must be large enough so that the above formula be valid.

If the transition point q_0 is real, but lies in the classically inaccessible region, then the above formula gives the transmission coefficient for penetration through a potential barrier, while if the transition point is complex, it solves the problem finding the over the barrier reflection coefficient. The $\sim$ sign in the above formula is used since the coefficient in front of the exponential is not determined. This can be determined by calculating the exact semi-classical wave functions. Generally, it is desirable to find the ratios of two different transitions so that this coefficient does not matter.

The physics of the tunnelling and the "over the barrier" reflection processes are very different. In the tunnelling process, the semi-classical analysis gives a transmission coefficient that is an exponentially small quantity with the corresponding reflection coefficient being unity. In contrast, in the "over the barrier" reflection process, just the reverse is obtained. The transmission coefficient is unity while the reflection coefficient is an exponentially small quantity. Both these processes will be encountered when the electric field is studied in different gauges.

We will now review the method of calculating the transmission and reflection coefficients for a typical quantum mechanical problem using the method of complex paths for a general potential $V(x)$.

1. Description of the method of complex paths

Consider the motion of a particle of mass m in a region characterised by the presence of a potential $V(x)$ in one space dimension. The problem is to calculate the transmission and reflection coefficients between two asymptotic regions labelled L and R where the semi-classical approximation to the exact wave function is valid. After identifying these regions and writing down the semi-classical wave functions, definite boundary conditions are imposed. The usual boundary conditions considered are that in one region, say L, the wavefunction is a superposition of an incident wave and a reflected wave while in the second region R, the wave function is just a transmitted wave. Then, a complex path (in the plane of the now complex variable x) is identified from R to L such that (a) all along the path the semiclassical ansatz is valid and (b) the reflected wave is exponentially greater than the incident wave at least in the latter part of the path near the region L. The transmitted wave is then moved along the path to obtain the reflected wave and thus the amplitude of reflection is identified in terms of the transmission amplitude. Having done this, the normalisation condition is used *i.e.* the sum of the modulus square of the transmission and reflection amplitudes should equal unity, to determine the exact values of the transmission and reflection coefficients.

For a given potential, the turning points q_0 (or transition points) are given by solving the equation

$$p(q_0) = \sqrt{2m(E - V(q_0))} = 0 \tag{4}$$

where $p(x)$ is the classical momentum of the particle and E is the energy of the particle. In general, q_0 is complex. At these points, the semi-classical ansatz is not valid since the momentum is zero. Further, the potential can possess singularities. At these points too, the semi-classical approximation is invalid. Therefore the contour between the two regions should be chosen to be far away from such points. In general the contour will enclose them. Therefore, the relation between the transmission and reflection amplitudes is determined by taking into account the turning points and the singularities of the potential.

The Schrodinger equation to determine the wave function ψ of the particle is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V(x))\psi. \tag{5}$$

Referring to [10], the semi-classical wave function, in the classically allowed region where $E > V(x)$, is given by the formula

$$\psi = C_1 p^{-1/2} \exp\left(\frac{i}{\hbar}\int p(x)dx\right) + C_2 p^{-1/2} \exp\left(-\frac{i}{\hbar}\int p(x)dx\right) \tag{6}$$

while in the classically inaccessible regions of space where $E < V(x)$, the function $p(x)$ is purely imaginary and the wave function is now given by the relation

$$\psi = C_1 |p|^{-1/2} \exp\left(-\frac{1}{\hbar}\int |p(x)|dx\right) + C_2 |p|^{-1/2} \exp\left(\frac{1}{\hbar}\int |p(x)|dx\right). \tag{7}$$

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The condition on the potential for semi-classicality of the wave function to be valid is

$$\left| \frac{d}{dx} \left(\frac{\hbar}{p(x)} \right) \right| \ll 1, \tag{8}$$

or, in another form,

$$\frac{m\hbar|F|}{|p|^3} \ll 1, \qquad F = -\frac{dV}{dx}. \tag{9}$$

It ought to be noted that the accuracy of the semi-classical approximation is not such as to allow the superposition of exponentially small terms over exponentially large ones. Therefore, it is inapplicable in general to retain both terms in Eqn. (7). We will consider a few cases of interest in this paper and refer the reader to [10] for a exhaustive discussion along with suitable illustrative examples.

Consider the case in which the semi-classical condition (9) holds in the regions $x \to \pm\infty$. As $x \to -\infty$, the wave function is assumed to be a superposition of incident and reflected waves and is written in the form,

$$\psi = p^{-1/2} \exp\left(\frac{i}{\hbar} \int p(x)dx \right) + C_2 p^{-1/2} \exp\left(-\frac{i}{\hbar} \int p(x)dx \right) \tag{10}$$

where the incident wave has unit amplitude while the reflected wave has amplitude given by C_2. As $x \to +\infty$, the wave function is assumed to be a right moving travelling wave,

$$\psi = \frac{C_3}{\sqrt{p}} \exp\left(\frac{i}{\hbar} \int pdx \right) \tag{11}$$

The method of complex paths can now be applied on the function (11). The contour is chosen either in the upper or lower complex plane such that the reflected wave is always exponentially greater than the incident wave along that part of the path near the region $x \to -\infty$. If this is satisfied along one of the contours then C_2 is determined in terms of C_3. To carry out the above procedure however, the exact semi-classical wave functions as $x \to \pm\infty$ have to be determined. This will be done explicitly later for the relevant cases.

A different case arises when the semi-classical ansatz holds in the vicinity of the origin $x = 0$ rather than at $x = \pm\infty$. The boundary conditions assumed in this case are the same as above with the condition $x \to \infty$ replaced by $x \to 0^+$ and $x \to -\infty$ by $x \to 0^-$. Here, the required contour is about the origin and is chosen to be small. But it must still be large enough so that the reflected wave is much larger than the incident wave along the latter part of the contour near the region $x < 0$.

In the above cases the method of complex paths gives the exact transmission and reflection amplitudes. But, in certain cases it is enough to assume that the transmission and incident amplitudes are equal to unity while the reflection amplitude is exponentialy damped and consequently very small. Here, the "over the barrier" reflection coefficient for energies large enough so that the reflection coefficient is exponentially small, has to be determined. In this case, the condition $E > V(x)$ is always satisfied. Therefore, the transition point q_0 at which the particle reverses its direction is the complex root of the equation $V(q_0) = E$. Let q_0 lie in the upper complex plane for definiteness. Now, the amplitudes of the incident and transmitted waves are equal (both are set to unity within exponential accuracy). To calculate the reflection coefficient, the relation between the wave functions far to the right of the barrier and far to the left of the barrier must be determined. The transmitted wave can be written in the form

$$\psi_T = \frac{1}{\sqrt{p}} \exp\left(\frac{i}{\hbar} \int_{x_1}^{x} pdx \right) \tag{12}$$

where x_1 is any point on the real axis. We follow the variation of ψ_T along a path C in the upper complex plane which encloses the turning point q_0. The latter part of this path must lie far enough to the left of q_0 so that the error in the semi-classical incident wave is less than the required small reflected wave. Passage around q_0 only causes a change in the sign of the root $\sqrt{E - V(x)}$ and after returning to the real axis, the function ψ_T becomes the reflected wave ψ_R. Going around a complex path in the lower complex plane converts ψ_T into the incident wave. Since the amplitudes of the incident and transmitted waves may be regarded as equal, the required reflection coefficient is given by

$$R = \left| \frac{\psi_R}{\psi_T} \right|^2 = \exp\left(-\frac{2}{\hbar} \mathrm{Im} \int_C pdx \right) \tag{13}$$

Now we can deform the contour in any way provided it still encloses the point q_0. In particular, the contour can be deformed to go from x_1 to q_0 and back. This gives

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$$R = \exp\left(-\frac{4}{\hbar}\mathrm{Im}\int_{x_1}^{q_0} p\,dx\right). \tag{14}$$

Since $p(x)$ is real everywhere, the choice of x_1 on the real axis is immaterial. The above formula determines the above the barrier reflection coefficient. It must be emphasised that to apply the above formula the exponent must be large so that $1 - R$ is very nearly equal to unity.

Finally consider a situation where the amplitudes of the reflection and incident wave are equal. The transmission coefficient is now an exponentially small quantity. This case corresponds to the standard tunnelling process. The problem is characterised by the presence of real turning points between which lies the classically forbidden region where the energy $E < V(x)$. For definiteness, let there be two real turning points labelled q_- and q_+. The potential $V(x)$, in the immediate vicinity of the turning points of the barrier, is assumed to be of the form

$$E - V(x) \approx F_0(x - q_{\pm}), \qquad F_0 = -\left.\frac{dV}{dx}\right|_{x=q_{\pm}} \tag{15}$$

This assumption is equivalent to saying that the particle, near the turning points, moves in a homogeneous field. With this assumption, the amplitude of transmission C_3 is given by (refer to Ref. [10] page 181)

$$C_3 = \exp\left(-\frac{1}{\hbar}\int_{q_-}^{q_+} |p(x)|\,dx\right) \tag{16}$$

The transmission coefficient is then given by $|C_3|^2$. The above formula holds only when the exponent is large. In the derivation above, we have assumed that the semi-classical condition holds across the entire barrier except in the immediate vicinity of the turning points. In general, however, the semi-classical condition need not hold over the entire extent of the barrier. The potential, for example, could drop steeply enough so that Eqn. (15) is not valid. In these cases, the exact semi-classical equations have to be determined before applying the method of complex paths. The cases encountered in this paper all satisfy Eqn. (15).

We now apply the above results to two potentials. The first is the well known inverted harmonic oscillator potential $V_1(x) = -g_1 x^2$ with $g_1 > 0$ while the other is $V_2(x) = -g_2/x^2$ with $g_2 > 0$. The first potential arises when the propagation of a scalar field in a constant electric field background is studied. The second potential arises when the propagation of a scalar field in Schwarzchild-like spacetimes is considered in the vicinity of the horizon.

2. Application to the potential $V_1(x) = -g_1 x^2$

Consider the potential given by

$$V_1(x) = -g_1 x^2 \tag{17}$$

where $g_1 > 0$ is a constant. This potential is the inverted harmonic oscillator potential and is discussed extensively in many places (see for example [6], [10]). We will follow the semi-classical treatment given in Ref. [10] and briefly review the calculation of the reflection and transmission coefficients for both the tunneling and over the barrier reflection cases.

The semi-classicality condition (9) for the above potential is,

$$\left|\frac{\hbar g_1 x}{\sqrt{2m}[E_1 + g_1 x^2]^{3/2}}\right| \ll 1 \tag{18}$$

where m is the mass of the particle and E_1 is its energy. The above condition definitely holds for large enough $|x|$ and for any value of E_1, either positive or negative. Therefore the motion of a particle moving under such a potential is semi-classical for large enough $|x|$ and hence holds as $x \to \pm\infty$.

Since the motion is semi-classical for large $|x|$, we can expand the momentum $p(x)$ as,

$$p(x) = \sqrt{2m\left(E_1 + g_1 x^2\right)} \approx \sqrt{2mg_1}\left(x + \frac{E_1}{2g_1 x}\right) \tag{19}$$

Using Eqn. (19), the semi-classical wave functions can be written as follows with the following boundary conditions. As $x \to \infty$, we assume that the wave function is a right moving travelling wave ψ_R while as $x \to -\infty$, it is a superposition of an incident wave of unit amplitude and a reflected wave given by ψ_L. Therefore, we have,

$$\psi_R = C_3 \xi^{i\varepsilon_1 - \frac{1}{2}} e^{i\xi^2/2} \qquad\qquad (\xi \to +\infty) \tag{20}$$

$$\psi_L = (-\xi)^{-i\varepsilon_1 - \frac{1}{2}} e^{-i\xi^2/2} + C_2 (-\xi)^{i\varepsilon_1 - \frac{1}{2}} e^{i\xi^2/2} \qquad (\xi \to -\infty) \tag{21}$$

where we have made the definitions

$$\xi = \left(\frac{2mg_1}{\hbar^2}\right)^{1/4} x \quad ; \quad \varepsilon_1 = \frac{1}{\hbar}\sqrt{\frac{m}{2g_1}} E_1 \tag{22}$$

Following the variation of Eqn. (20) around a semi-circle of large radius ρ in the *upper* half plane of the now complex variable ξ, we obtain

$$C_2 = -iC_3 \exp\left(-\pi\varepsilon_1\right) \tag{23}$$

The conservation of particles is expressed by the condition that,

$$|C_3|^2 + |C_2|^2 = 1 \tag{24}$$

From Eqn. (23) and Eqn. (24), the transmission coefficient is,

$$T = |C_3|^2 = \frac{1}{1 + e^{-2\pi\varepsilon_1}} = \frac{1}{1 + e^{-\frac{1}{\hbar}\pi\sqrt{2m/g_1}E_1}} \tag{25}$$

while the reflection coefficient is

$$R = |C_2|^2 = \frac{e^{-\frac{1}{\hbar}\pi\sqrt{2m/g_1}E_1}}{1 + e^{-\frac{1}{\hbar}\pi\sqrt{2m/g_1}E_1}} \tag{26}$$

Note that the passage through the *lower* half complex plane to determine C_2 is unsuitable since on the part of the path $-\pi < \phi < -\pi/2$, where ϕ is the phase of the complex variable ξ, the incident wave (first term in Eqn. (21)) is exponentially large compared with the reflected wave. The above formula holds for all energies E_1. This is because, even for negative energies, the semi-classical wave functions given in Eqns. (20, 21) are exactly the same with the boundary conditions being fully satisfied.

If E_1 is large and negative, Eqn. (25) gives $T \approx e^{-\pi\sqrt{2m/g_1}|E_1|/\hbar}$ and thus $R \sim 1$. This is in accordance with the formula in Eqn. (16). To apply Eqn. (16) it is necessary to calculate the turning points first. The real turning points are $q_0 = -\sqrt{|E_1|/g_1}$ and $q_1 = \sqrt{|E_1|/g_1}$. Therefore,

$$
\begin{aligned}
C_3 &= \exp\left(-\frac{1}{\hbar}\int_{q_0}^{q_1} |p(x)| dx\right) \\
&= \exp\left(-\frac{1}{\hbar}\sqrt{2mg_1}\int_{q_0}^{q_1}\left|\sqrt{x^2 - \frac{|E_1|}{g_1}}\right| dx\right) \\
&= \exp\left(-\frac{1}{2\hbar}\pi\sqrt{2m/g_1}|E_1|\right)
\end{aligned}
\tag{27}
$$

This gives the same answer.

We can calculate the over the barrier reflection coefficient using Eqn. (14) for E_1 large and *positive*. The turning points now are given by solving the equation $p(q_0) = 0$ with the condition that $E_1 > V_1(x)$ always. Since $E_1 > 0$, the turning points are $q_0 = \pm i\sqrt{E_1/g_1}$. Choosing the positive sign for q_0 and setting $x_1 = 0$, the integral in Eqn. (14) is evaluated as follows:

$$
\begin{aligned}
\int_0^{q_0} p(x) dx &= \sqrt{2mg_1}\int_0^{q_0}\sqrt{E_1/g_1 + x^2} \\
&= i\sqrt{2m/g_1}E_1\int_0^1 \sqrt{1 - y^2} = \frac{1}{4}i\pi\sqrt{2m/g_1}E_1
\end{aligned}
\tag{28}
$$

Therefore,

$$R = \exp\left(-\frac{1}{\hbar}\pi\sqrt{2m/g_1}E_1\right) \tag{29}$$

The above formula can also be obtained directly from Eqn. (26) by neglecting the exponential term compared to unity which means that the energy has to be large.

Consider the potential given by

$$V_1(x) = -\frac{g_2}{x^2} \tag{30}$$

where g_2 is a positive constant. The potential has a singularity at the origin. This potential arises when the effective Schrodinger equation is calculated for Schwarzchild-like spacetimes. This aspect will be dealt with in later sections.

The semi-classical condition (9) for this potential takes the form

$$\left| \frac{\hbar g_2}{\sqrt{2m}} \frac{1}{[E_2 x^2 + g_2]^{3/2}} \right| \ll 1 \tag{31}$$

where E_2 is the energy. It is clear that the above relation holds for large $|x|$. It also holds for small $|x|$ if $\sqrt{2mg_2} \gg \hbar$. Notice that the quasi-classicality condition for small $|x|$ is independent of the sign and magnitude of the energy E_2. For this potential, we will be concerned only with the small $|x|$ behaviour in contrast with the potential V_1. Since the motion is semi-classical for small $|x|$, we expand the momentum $p(x)$ as

$$p(x) = \sqrt{2m\left(E_2 + \frac{g_2}{x^2}\right)} \approx \frac{\sqrt{2mg_2}}{x} + \sqrt{\frac{m}{2g_2}} E_2 x \tag{32}$$

Notice the similarity between Eqn. (19) and Eqn. (32).

We will calculate the over the barrier reflection coefficient with $E_2 > 0$ and small, which will be of interest later. Using the expansion in Eqn. (32), the semi-classical wave functions with the following boundary conditions, namely, that for $x > 0$ the wave function is a right moving travelling wave while it is a superposition of an incident wave of unit amplitude and reflected wave for $x < 0$, are

$$\psi_R = C_3 \xi^{i\varepsilon_2 + \frac{1}{2}} e^{i\xi^2/2} \qquad\qquad (\xi > 0) \tag{33}$$

$$\psi_L = (-\xi)^{-i\varepsilon_2 + \frac{1}{2}} e^{-i\xi^2/2} + C_2(-\xi)^{i\varepsilon_2 + \frac{1}{2}} e^{i\xi^2/2} \qquad (\xi < 0) \tag{34}$$

where we have made the definitions

$$\xi = \left(\frac{mE_2^2}{2g_2}\right)^{1/4} x \qquad ; \qquad \varepsilon_2 = \frac{\sqrt{2mg_2}}{\hbar} \tag{35}$$

Following the variation of Eqn. (33) around an small semi-circle of radius $\rho < \sqrt{g_2/|E_2|}$ (in contrast to the potential V_1 where the radius ρ was large) in the *upper* half complex plane, we obtain,

$$C_2 = C_3 \exp\left(-\pi\varepsilon_2 + \frac{i\pi}{2}\right) \tag{36}$$

Setting $T = |C_3|^2 = 1$ and $R = |C_2|^2$, we finally obtain,

$$R = Te^{-2\pi\varepsilon_2} = Te^{-\frac{1}{\hbar}2\pi\sqrt{2mg_2}} \tag{37}$$

Using the normalisation condition $R + T = 1$, we obtain,

$$T = \frac{1}{1 + e^{-\frac{1}{\hbar}2\pi\sqrt{2mg_2}}} \quad \text{and} \quad R = \frac{e^{-\frac{1}{\hbar}2\pi\sqrt{2mg_2}}}{1 + e^{-\frac{1}{\hbar}2\pi\sqrt{2mg_2}}} \tag{38}$$

Notice that the above result is independent of the energy E_2 and hence holds for $E_2 < 0$ too. For small $|x|$, the lack of dependence on E_2 is not too surprising since the contour is such that it is not too close to the real turning points $q_0 = \pm\sqrt{g_2/|E_2|}$. Anyway, when $E_2 \sim 0^+$, ρ is "large" and therefore the contour is chosen to lie in the upper complex plane for the same reason as given in the analysis of the potential V_1 in the previous section.

Now, we will derive the result in Eqn. (37) using Eqn. (13). The complex turning points q_0 are the roots of the equation $E_2 = -g_2/q_0^2$ where $E_2 > 0$ and therefore, the turning points are $q_0 = \pm i\sqrt{g_2/E_2} = \pm ip_0$. Hence, we have to evaluate the integral,

$$\int_C p\,dx = \sqrt{2mE_2} \int_C \sqrt{1 + \frac{p_0^2}{x^2}}\,dx \tag{39}$$

where the contour C encircles the point $x = ip_0$ in the upper half complex plane. However, since there is a singularity at the origin, we cannot deform the contour as was done when deriving Eqn. (14). Therefore, as a means of regularisation, we modify the potential to

$$V_{\text{mod}}(x) = -\frac{g_2}{x^2 + \epsilon^2} \tag{40}$$

where the limit $\epsilon \to 0$ must be taken at the end of the calculation. The turning points for the modified potential are $x_{\text{mod}} = \pm i\sqrt{\epsilon^2 + g_2/E_2}$ while the poles of the modified potential are at $x = \pm i\epsilon < x_{\text{mod}}$. Even in this case, there is a singularity on the path of integration which contributes to the integral rather than the turning point. Therefore, integrating upto $+i\epsilon$ using the modified potential and back, we obtain,

$$\begin{aligned}
\int_C p\,dx &= \lim_{\epsilon \to 0} 2\sqrt{2mE_2} \int_0^{i\epsilon} \sqrt{1 + \frac{p_0^2}{x^2 + \epsilon^2}}\,dx \\
&= \lim_{\epsilon \to 0} 2i\sqrt{2mE_2}\,\epsilon \int_0^1 dy \sqrt{1 + \frac{p_0^2/\epsilon^2}{1 - y^2}} \\
&\approx 2i\sqrt{2mE_2}\,p_0 \int_o^1 \frac{dy}{\sqrt{1 - y^2}} \\
&= i\pi\sqrt{2mE_2}\,p_0 = i\pi\sqrt{2mg_2}
\end{aligned} \tag{41}$$

We therefore recover the result given in Eqn. (37). From the above calculation it is clear that, due to the singularity at the origin, the reflection coefficient has no contribution from the turning point at all.

III. PARTICLE PRODUCTION IN AN UNIFORM ELECTRIC FIELD

We will now study a system consisting of a minimally coupled scalar field Φ propagating in flat spacetime in an uniform electric field background. We consider two gauges, one a time dependent gauge while the other is a space dependent gauge and show how the tunnelling interpretation can be used to obtain the standard gauge invariant result in each case. We will not derive the standard result here.

A. Time dependent gauge

The four vector potential A^μ giving rise to a constant electric field in the x direction is assumed to be of the form

$$A^\mu = (0, -E_0 t, 0, 0) \tag{42}$$

The electric field is $\mathbf{E} = E_0 \hat{\mathbf{x}}$. The minimally coupled scalar field Φ propagating in flat spacetime, satisfies the Klein-Gordon equation,

$$\left((\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2 \right) \Phi = 0 \tag{43}$$

where m is the mass and q is the charge of the field. The mode functions of Φ can be expressed in the form $\Phi(t, \mathbf{x}) = f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$ where $f_{\mathbf{k}}(t)$ satisfies the equation,

$$\frac{d^2}{dt^2} f_{\mathbf{k}} + \left[m^2 + k_\perp^2 + (k_x + qE_0 t)^2 \right] f_{\mathbf{k}} = 0 \; ; \quad \mathbf{k}_\perp = (k_y, k_z). \tag{44}$$

Introducing the variables,

$$\tau = \sqrt{qE_0}\,t + \left(\omega/\sqrt{qE_0} \right) \; ; \; \lambda = (k_\perp^2 + m^2)/qE_0 \tag{45}$$

we obtain the equation,

$$-\frac{d^2}{d\tau^2}f_{\mathbf{k}} - \tau^2 f_{\mathbf{k}} = \lambda f_{\mathbf{k}} \tag{46}$$

The above equation is essentially a Schrodinger equation in an inverted oscillator potential with a positive "energy" λ. Since the energy is positive, the problem is essentially an over the barrier reflection problem. Using the results of section (II A 2), we can calculate the reflection and transmission coefficients exactly as

$$R = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}} \quad ; \quad T = \frac{1}{1 + e^{-\pi\lambda}} \tag{47}$$

where we have put $\hbar = 2m = \omega_0/2 = 1$ and set $a^2 = \lambda$ in Eqns. (25, 26). To identify the Bogoliubov coefficients α_λ and β_λ, we recast the normalisation condition $R + T = 1$ in the form,

$$\frac{1}{T} - \frac{R}{T} = 1 \tag{48}$$

and then identify $|\beta_\lambda|^2$ with R/T and $|\alpha_\lambda|^2$ with $1/T$. Therefore, the Bogoliubov coefficients are given by,

$$\begin{aligned}
|\beta_\lambda|^2 &= e^{-\pi\lambda} = e^{-\pi(k_\perp^2 + m^2)/qE_0} \\
|\alpha_\lambda|^2 &= e^{-\pi\lambda} + 1 = e^{-\pi(k_\perp^2 + m^2)/qE_0} + 1.
\end{aligned} \tag{49}$$

Now, the transmission and reflection coefficients are time reversal invariant. They are dependent only on the energy (magnitude and sign) and for symmetric potentials, independent of the direction in which the boundary conditions are applied. To obtain a dynamical picture of particle production, we have to interpret these quantities suitably. In the present case, the following interpretation seems adequate. A purely positive frequency wave with amplitude square T in the infinite past, $t \to -\infty$, evolves into a combination of positive and negative frequency waves in the infinite future $t \to \infty$ with the negative frequency waves having an amplitude square R and the positive frequency waves having an amplitude unity. The quantity R/T determines the overlap between the negative frequency modes in the distant future and the positive frequency modes in the distant past (the notation here differs from the treatment given in [3], [2]). This is identified with the modulus square of the Bogoliubov coefficient β_λ which is the particle production per mode λ. Using the normalisation condition satisfied by the Bogoliubov coefficients, $|\alpha_\lambda|^2 - |\beta_\lambda|^2 = 1$, $|\alpha_\lambda|^2$ can be calculated to be $1/T$. Once the Bogoliubov coefficients have been identified, the effective lagrangian can be easily calculated. This derivation will not be repeated here. We refer the reader to Ref. [2] and Ref. [3] for the explicit calculation.

Note that the particular interpretation given in this case is due to its similarity with the more rigourous calculation by quantum field theoretic methods. In the next section, where we discuss the space dependent gauge, we will be forced to adopt a different interpretation in order to identify particle production.

B. Space dependent gauge

The four vector potential A^μ giving rise to a constant electric field in the x direction is now assumed to be of the form

$$A^\mu = (-E_0 x, 0, 0, 0) \tag{50}$$

The electric field is $\mathbf{E} = E_0\hat{\mathbf{x}}$. The field Φ satisfies Eqn. (43) as before. Substituting for the potential A^μ from Eqn. (50) into Eqn. (43), we obtain,

$$\left(\partial_t^2 - \nabla^2 - 2iqE_0 x\partial_t - q^2 E_0^2 x^2 + m^2\right)\Phi = 0 \tag{51}$$

We write Φ in the form

$$\Phi = e^{-i\omega t} e^{ik_y y + ik_z z} \phi(x) \tag{52}$$

and obtain the differential equation satisfied by ϕ as

$$\frac{d^2\phi}{dx^2} + \left((\omega + qE_0 x)^2 - k_\perp^2 - m^2\right)\phi = 0 \tag{53}$$

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where we have used the notation $k_\perp^2 = k_y^2 + k_z^2$. Making the following change of variables

$$\rho = \sqrt{qE_0}x + \left(\omega/\sqrt{qE_0}\right) \quad ; \quad \lambda = (k_\perp^2 + m^2)/qE_0 \tag{54}$$

into the differential equation for ϕ, it reduces to the form,

$$-\frac{d^2\phi}{d\rho^2} - \rho^2\phi = -\lambda\phi \tag{55}$$

In this form, we see that the above differential equation has the form of an effective Schrodinger equation with an inverted harmonic oscillator potential and an negative energy $-\lambda$. If we apply the results of section (II A 2), we obtain the result for tunneling through the barrier. Following the treatment in Ref. [3] and using the results of section (II A 2) we can calculate the reflection and transmission coefficients exactly as

$$R = \frac{e^{\pi\lambda}}{1 + e^{\pi\lambda}} \quad ; \quad T = \frac{1}{1 + e^{\pi\lambda}} \tag{56}$$

where we have put $\hbar = 2m = \omega_0/2 = 1$ and set $a^2 = \lambda$ in Eqns. (25, 26). We cast the renormalisation condition $R + T = 1$ in the form

$$\frac{1}{R} - \frac{T}{R} = 1 \tag{57}$$

and then identify the rate of particle production per mode with T/R. The interpretation of particle production using the tunnelling picture now proceeds as follows. A right moving travelling wave of amplitude square $1/R$ is incident on the potential. A fraction T/R is transmitted through it and a wave of unit amplitude is scattered back. The tunnelling probability, which is T/R, is interpreted as the rate at which particles are being produced by the background electric field. This matches exactly with the expression for $|\beta_\lambda|^2$ given in Eqn. (49). With this interpretation, we recover the usual gauge independent result.

In summary, it is seen that by a judicious choice of interpretation of the transmission and reflection coefficients in each of the two gauges, the standard gauge invariant result can be obtained. In fact, the tunnelling interpretation can also be used for gauges of the form

$$A^\mu = (-E_0x \pm E_1t, 0, 0, 0) \quad ; \quad A^\mu = (0, -E_0t \pm E_1x, 0, 0) \tag{58}$$

where the condition $|E_1| \neq |E_0|$ holds strictly. Depending on the magnitude of E_1, the problem reduces to either an "over the barrier reflection" or a "tunnelling through the barrier" process. The transmission and reflection coefficients do not depend on E_1 as it ought to be. But the tunnelling interpretation is not always valid. For instance, setting $|E_1| = |E_0|$ in the above gauges reduces the problem to a first order differential equation instead of a second order one. (This case will be considered later in a future publication.) Further, one can show that even in a time independent magnetic field, solving the effective Schrodinger equation gives non-zero transmission and reflection coefficients [11]. Since these coefficients are non-zero, applying the tunnelling picture implies particle production which is contrary to the result obtained by Schwinger in [1]. Hence, the tunnelling picture does not always produce consistent results. In the next section, we look at Schwarzschild-like spacetimes with a horizon. Here too, we construct the tunnelling picture near the horizon in order to explain particle production. We shall see that such a picture does not produce the correct results.

IV. PARTICLE PRODUCTION IN SPACETIMES WITH HORIZON

Hawking's result that a black hole radiates is essentially a semi-classical result. The thermal radiation results because of the presence of a horizon in the spacetime structure. We will review briefly the conventional derivation of the thermal radiation using path integrals. Consider a patch of spacetime, which in a suitable co-ordinate system, has one of the following forms (we assume $c = 1$):

$$ds^2 = B(r)dt^2 - B^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{59}$$

or

$$ds^2 = B(x)dt^2 - B^{-1}(x)dx^2 - dy^2 - dz^2 \tag{60}$$

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where $B(r)$ and $B(x)$ are functions of r and x respectively. The horizon in the above spacetimes is indicated by the surface $r = r_0$ ($x = x_0$) at which $B(r)$ ($B(x)$) vanishes. We further assume that $B'(r) = dB/dr$ ($B'(x) = dB/dx$) is finite and non-zero at the horizon. Co-ordinate systems of the form (59) can be introduced in parts of the Schwarzschild and de-Sitter spacetimes while that of the form (60) with the choice $B(x) = 1 + 2gx$ represents a Rindler frame in flat spacetime. Given the co-ordinate system of (59) say, in some region R, we first verify that there is no physical singularity at the horizon, which in the case of the Schwarzschild black hole, is at the co-ordinate value $r_0 = 2M$ where M is the mass of the black hole. Having done that, we extend the geodesics into the past and future and arrive at two further regions of the manifold not originally covered by the co-ordinate system in (59) (the Kruskal extension). It is now possible to show that the probability for a particle with energy E to be lost from the region R in relation to the probability for a particle with energy E to be gained by the region R is given by the relation

$$P_{\text{loss}} = P_{\text{gain}} e^{-\beta E} \tag{61}$$

where $\beta = 8\pi M$. This is equivalent to assuming that the region R is bathed in radiation at temperature β^{-1}. In the derivation given in the paper by Hartle and Hawking [4], thermal radiation is derived using the semiclassical kernel by an analytic continuation in the time co-ordinate t to complex values and it was shown that the probability of emission (loss) from the past horizon was related to absorption (gain) into the future horizon by the relation (61).

Since all the physics is contained in the (t, r) plane (or the (t, x) plane), we will discuss Hawking radiation in $1 + 1$ dimensions in the following sections and later show in the appendix that the results generalise naturally to $3 + 1$ dimensions. We first derive the semi-classical result in the (r, t) (or (x, t)) plane by applying a certain prescription to bypass the singularity encountered at the horizon. After this, we reduce the problem of the Klein-Gordon field propagating in the Schwarzschild spacetime to an effective Schrodinger problem in (1+1) dimensions and rederive the semi-classical result by using the same prescription on the semi-classical wave functions. Then the results of section (II) are used to compute the ratio of the transmission and reflection coefficients and we show that this ratio satisfies an equation of the form

$$R = T e^{-\frac{1}{2}\beta E} \tag{62}$$

where $\beta^{-1} = (8\pi M)^{-1}$ is the expected Hawking temperature. This can be interpreted to mean thermal radiation in exactly the same way as Eqn. (61) but with a temperature *twice* the expected value. This proves that the tunnelling picture, when naively applied, does not give the same result as the semi-classical analysis does. A possible qualitative explanation of this result will also be given.

A. Hawking Radiation in 1+1 dimensions

Consider a certain patch of spacetime in (1+1) dimensions which in a suitable co-ordinate system has the line element, (with $c = 1$)

$$ds^2 = +B(r)dt^2 - B^{-1}(r)dr^2 \tag{63}$$

where $B(r)$ is an arbitrary function of r. We assume that the function $B(r)$ vanishes at some $r = r_0$ with $B'(r) = dB/dr$ being finite and nonzero at r_0. The point $r = r_0$ indicates the presence of a horizon. It can be easily verified that no physical singularity exists at the horizon since the only component of the curvature tensor $\mathcal{R}_{trtr} = -(1/2)(d^2 B(r)/dr^2)$ does not become infinite at the horizon. Therefore, near the horizon, we may expand $B(r)$ as

$$B(r) = B'(r_0)(r - r_0) + \mathcal{O}[(r - r_0)^2] = R(r_0)(r - r_0). \tag{64}$$

We now consider a minimally coupled scalar field Φ with mass m_0 propagating in the spacetime represented by the metric (63). The equation satisfied by the scalar field is,

$$\left(\Box + \frac{m_0^2}{\hbar^2}\right)\Phi = 0 \tag{65}$$

where the $\Box$ operator is to be evaluated using metric (63). Expanding the LHS of equation (65), one obtains,

$$\frac{1}{B(r)}\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial}{\partial r}\left(B(r)\frac{\partial \Phi}{\partial r}\right) = -\frac{m_0^2}{\hbar^2}\Phi. \tag{66}$$

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The semiclassical wave functions satisfying the above equation are obtained by making the standard ansatz for Φ which is,

$$\Phi(r,t) = \exp\left[\frac{i}{\hbar}S(r,t)\right] \tag{67}$$

where S is a function which will be expanded in powers of $\hbar$. Substituting into the wave equation (66), we obtain,

$$\left[\frac{1}{B(r)}\left(\frac{\partial S}{\partial t}\right)^2 - B(r)\left(\frac{\partial S}{\partial r}\right)^2 - m_0^2\right] +$$
$$\left(\frac{\hbar}{i}\right)\left[\frac{1}{B(r)}\frac{\partial^2 S}{\partial t^2} - B(r)\frac{\partial^2 S}{\partial r^2} - \frac{dB(r)}{dr}\frac{\partial S}{\partial r}\right] = 0 \tag{68}$$

Expanding S in a power series of $(\hbar/i)$,

$$S(r,t) = S_0(r,t) + \left(\frac{\hbar}{i}\right)S_1(r,t) + \left(\frac{\hbar}{i}\right)^2 S_2(r,t)\ldots \tag{69}$$

and substituting into Eqn (68) and neglecting terms of order $(\hbar/i)$ and greater, we find to the lowest order,

$$\frac{1}{B(r)}\left(\frac{\partial S_0}{\partial t}\right)^2 - B(r)\left(\frac{\partial S_0}{\partial r}\right)^2 - m_0^2 = 0 \tag{70}$$

Eqn (70) is just the Hamilton-Jacobi equation satisfied by a particle of mass m_0 moving in the spacetime determined by the metric (63). The solution to the above equation is

$$S_0(r,t) = -Et \pm \int^r \frac{dr}{B(r)}\sqrt{E^2 - m_0^2 B(r)} \tag{71}$$

where E is a constant and is identified with the energy. Notice that in the case of $m_0 = 0$, Eqn (70) can be exactly solved with the solution

$$S_{m_0=0}(r,t) = F_1(t - r*) + F_2(t + r*) \tag{72}$$

where the "tortoise" co-ordinate $r*$ is defined by

$$r* = \int \frac{dr}{B(r)}, \tag{73}$$

and F_1 and F_2 are arbitrary functions. If F_1 is chosen to be $F_1 = -Et + Er*$ and F_2 chosen to be $F_2 = -Et - Er*$, then it is clear that the solution given in Eqn (72) is the same as that in Eqn (71) with m_0 set to zero. Therefore, in the case $m_0 = 0$, the semiclassical ansatz is exact. In the following analysis we will specialise to the case $m_0 = 0$ for simplicity. The case $m_0 \neq 0$ will be considered later. The essential results do not change in any way.

Knowing the semiclassical wave function, the semiclassical kernel $K(r_2, t_2; r_1, t_1)$ for the particle to propagate from (t_1, r_1) to (t_2, r_2) in the saddle point approximation can be written down immediately.

$$K(r_2, t_2; r_1, t_1) = N \exp\left(\frac{i}{\hbar}S(r_2, t_2; r_1, t_1)\right) \tag{74}$$

where S is the action functional satisfying the classical Hamilton-Jacobi equation in the massless limit and N is a suitable normalization constant. $S(r_2, t_2; r_1, t_1)$ is given by the relation

$$S(r_2, t_2; r_1, t_1) = S(2,1) = -E(t_2 - t_1) \pm E\int_{r_1}^{r_2}\frac{dr}{B(r)}. \tag{75}$$

The sign ambiguity (of the square root) is related to the "outgoing" ($\partial S/\partial r > 0$) or "ingoing" ($\partial S/\partial r < 0$) nature of the particle. As long as points 1 and 2, between which the transition amplitude is calculated, are on the same side of the horizon (i.e. both are in the region $r > r_0$ or in the region $r < r_0$), the integral in the action is well defined

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and real. But if the points are located on opposite sides of the horizon then the integral does not exist due to the divergence of $B^{-1}(r)$ at $r = r_0$.

Therefore, in order to obtain the probability amplitude for crossing the horizon we have to give an extra prescription for evaluating the integral [3]. Since the point $B = 0$ is null we may carry out the calculation in Euclidean space or —equivalently—use an appropriate $i\epsilon$ prescription to specify the complex contour over which the integral has to be performed around $r = r_0$. The prescription we use is that we should take the contour for defining the integral to be an infinitesimal semicircle *above* the pole at $r = r_0$ for outgoing particles on the left of the horizon and ingoing particles on the right. Similarly, for ingoing particles on the left and outgoing particles on the right of the horizon (which corresponds to a time reversed situation of the previous cases) the contour should be an an infinitismal semicircle *below* the pole at $r = r_0$. Equivalently, this amounts to pushing the singularity at $r = r_0$ to $r = r_0 \mp i\epsilon$ where the upper sign should be chosen for outgoing particles on the left and ingoing particles on the right while the lower sign should be chosen for ingoing particles on the left and outgoing particles on the right. For the Schwarzchild case, this amounts to adding an imaginary part to the mass since $r_0 = 2M$.

Consider therefore, an outgoing particle ($\partial S/\partial r > 0$) at $r = r_1 < r_0$. The modulus square of the amplitude for this particle to cross the horizon gives the probability of emission of the particle. The contribution to S in the ranges $(r_1, r_0 - \epsilon)$ and $(r_0 + \epsilon, r_2)$ is real. Therefore, choosing the contour to lie in the upper complex plane,

$$S[\text{emission}] = -\lim_{\epsilon \to 0} \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{dr}{B(r)} + \text{ (real part)}$$
$$= \frac{i\pi E}{R(r_0)} + \text{(real part)} \tag{76}$$

where the minus sign in front of the integral corresponds to the initial condition that $\partial S/\partial r > 0$ at $r = r_1 < r_0$. For the sake of definiteness we have assumed $R(r_0)$ in Eqn (64) to be positive, so that $B(r) < 0$ when $r < r_0$. (For the case when $R < 0$, the answer has to be modified by a sign change.) The same result is obtained when an ingoing particle ($\partial S/\partial r < 0$) is considered at $r = r_1 < r_0$. The contour for this case must be chosen to lie in the lower complex plane. The amplitude for this particle to cross the horizon is the same as that of the outgoing particle due to the time reversal invariance symmetry obeyed by the system.

Consider next, an ingoing particle ($\partial S/\partial r < 0$) at $r = r_2 > r_0$. The modulus square of the amplitude for this particle to cross the horizon gives the probability of absorption of the particle into the horizon. Choosing the contour to lie in the upper complex plane, we get,

$$S[\text{absorption}] = -\lim_{\epsilon \to 0} \int_{r_0+\epsilon}^{r_0-\epsilon} \frac{dr}{B(r)} + \text{ (real part)}$$
$$= -\frac{i\pi E}{R(r_0)} + \text{(real part)} \tag{77}$$

The same result is obtained when an outgoing particle ($\partial S/\partial r > 0$) is considered at $r = r_2 > r_0$. The contour for this case should be in the lower complex plane and the amplitude for this particle to cross the horizon is the same as that of the ingoing particle due to time reversal invariance.

Taking the modulus square to obtain the probability P, we get,

$$P[\text{emission}] \propto \exp\left(-\frac{2\pi E}{R(r_0)}\right) \tag{78}$$

and

$$P[\text{absorption}] \propto \exp\left(\frac{2\pi E}{R(r_0)}\right) \tag{79}$$

so that

$$P[\text{emission}] = \exp\left(-\frac{4\pi E}{R(r_0)}\right) P[\text{absorption}]. \tag{80}$$

Now time reversal invariance implies that the probability for the emission process is the same as that for the absorption process proceeding backwards in time and *vice versa*. Therefore we must interpret the above result as saying that the probability of emission of particles is not the same as the probability of absorption of particles. In other words, if the horizon emits particles at some time with a certain emission probability, the probability of absorption of particles at

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the same time is different from the emission probability. This result shows that it is more likely for a particular region to gain particles than lose them. Further, the exponential dependence on the energy allows one to give a 'thermal' interpretation to this result. In a system with a temperature β^{-1} the absorption and emission probabilities are related by

$$P[\text{emission}] = \exp(-\beta E)P[\text{absorption}] \tag{81}$$

Comparing Eqn (81) and Eqn (80), we identify the temperature of the horizon in terms of $R(r_0)$. Eqn (80) is based on the assumption that $R > 0$. If $R < 0$ there will be a change of sign in the equation. Incorporating both the cases, the general formula for the horizon temperature is

$$\beta^{-1} = \frac{|R|}{4\pi} \tag{82}$$

For the Schwarzchild black hole,

$$B(r) = \left(1 - \frac{2M}{r}\right) \approx \frac{1}{2M}(r - 2M) + \mathcal{O}[(r - 2M)^2] \tag{83}$$

giving $R = (2M)^{-1}$, and the temperature $\beta^{-1} = 1/8\pi M$. For the de-Sitter spacetime,

$$B(r) = (1 - H^2 r^2) \approx 2H(H^{-1} - r) = -2H(r - H^{-1}) \tag{84}$$

giving $\beta^{-1} = H/2\pi$. Similarly for the Rindler spacetime

$$B(r) = (1 + 2gr) = 2g(r + (2g)^{-1}) \tag{85}$$

giving $\beta^{-1} = g/2\pi$. The formula for the temperature can be used for more complicated metrics as well and gives the same results as obtained by more detailed methods.

The prescription given for handling the singularity is analogous to the analytic continuation in time proposed by Hawking [4] to derive Black hole radiance. If one started out on the left of the horizon and went around the singularity $r = r_0$ by a 2π rotation instead of a rotation by π, it can be easily shown that it has the effect of taking the Kruskal co-ordinates (v, u) to $(-v, -u)$. A full rotation by 2π around the singularity can be split up into two parts to give the amplitude for emission and subsequent absorption of a particle with energy E. Since the amplitudes for the two processes are not the same in the presence of a horizon, one obtains the usual Hawking radiation given in Eqn (80) with the value of $R(r_0)$ being $(2M)^{-1}$. This process is similar to that given in [4] which relates the amplitudes involving the past and future horizons. In Hawking's paper, analytically continuing the time variable t to $t - 4Mi\pi$ takes the Kruskal co-ordinates (v, u) to $(-v, -u)$ and since the path integral kernel is analytic in a strip of $4Mi\pi$ below the real t axis, Hawking radiation is obtained by deforming the contour of integration appropriately.

When $m_0 \neq 0$, the validity of the semi-classical ansatz must be verified. To do this, consider the perturbative expansion (69). Retaining the terms of order $\hbar/i$ and neglecting higher order terms, one finds, upon substituting for S_0 given by the relation (71) and solving for S_1,

$$S_1 = -E_1 t \pm E E_1 \int \frac{dr}{B(r)} \frac{1}{\sqrt{E^2 - m_0^2 B(r)}} - \frac{1}{4}\ln(E^2 - m_0^2 B(r)) \tag{86}$$

where E_1 is a constant. From the above equation, it is seen that S_1 has a singularity of the same order as S_0 at $r = r_0$. When calculating the amplitude to cross the horizon, the contribution from the singular term just appears as a phase factor multiplying the semiclassical kernel and is inconsequential. The non-singular finite terms do contribute to the kernel but they contribute the same amount to $S[\text{emission}]$ and $S[\text{absorption}]$ and they do not affect the relation between the probabilities $P[\text{emission}]$ and $P[\text{absorption}]$. Subsequent calculation of the terms S_2, S_3, and so on, show that all these terms have a singularity at the horizon of the same order as that of S_0. Their contribution to the probability amplitude is just a set of terms multiplied by powers of $\hbar$ which can be neglected. From this we may conclude that the semiclassical ansatz, in the perturbative limit, is a valid one.

B. Reduction to an effective Schrodinger Problem in (1+1) dimensions

Consider the relativistic equation for the wave function Φ in Eqn (66). We include the mass m_0 here but later we shall see that it does not appear in the final reduced Schrodinger equation. We first set $\Phi = e^{-iEt/\hbar}\Psi(r)$ to obtain,

$$B(r)\frac{d^2\Psi}{dr^2} + \frac{dB(r)}{dr}\frac{d\Psi}{dr} + \left(\frac{E^2}{\hbar^2 B(r)} - \frac{m_0^2}{\hbar^2}\right)\Psi = 0 \tag{87}$$

We then make the substitution

$$\Psi(r) = P(r)Q(r) \ \text{ with } P(r) = \frac{1}{\sqrt{B(r)}} \tag{88}$$

to get the equation,

$$-\frac{d^2Q(r)}{dr^2} - \left[-\frac{B''(r)}{2B(r)} + \frac{(B'(r))^2}{4B^2(r)} + \frac{E^2}{\hbar^2 B^2(r)} - \frac{m_0^2}{\hbar^2 B(r)}\right]Q(r) = 0 \tag{89}$$

where B' denotes dB/dr and B'' denotes d^2B/dr^2. Near the horizon, we use the expansion of $B(r)$ given in Eqn (64). Neglecting terms of order $1/(r - r_0)$ as compared to terms of order $1/(r - r_0)^2$, we get,

$$-\frac{d^2Q(r)}{dr^2} - \frac{g}{(r - r_0)^2}Q(r) = 0 \qquad \text{where} \qquad g = \left(\frac{1}{4} + \frac{E^2}{\hbar^2 R^2}\right) \tag{90}$$

Notice that m_0 appears nowhere in the above equation. Very close to the horizon, the term containing the mass does not contribute significantly. Eqn (90) is therefore applicable to both massless and massive scalar particles. Making the simple transformation $x = (r - r_0)$, we finally obtain the effective Schrodinger equation for the system with $\hbar = 2m = 1$ with a potential $(-g/x^2)$.

$$-\frac{d^2Q(x)}{dx^2} - \frac{g}{x^2}Q(x) = 0 \tag{91}$$

This potential is symmetric about the origin $x = 0$. We now include an "energy" $\widetilde{E}$ (not to be confused with the energy E of the field in the original relativistic system) to obtain,

$$-\frac{d^2Q(x)}{dx^2} - \frac{g}{x^2}Q(x) = \widetilde{E}Q(x). \tag{92}$$

The energy $\widetilde{E}$, which can be positive or negative, has been included in order to consider fully the properties of the potential $-g/x^2$. We will finally be interested in the case $\widetilde{E} = 0$ which is the case of interest. (The energy spectrum is continuous for all values of $\widetilde{E}$ which, for $\widetilde{E} < 0$, is surprising since for energies less than the potential energy, the spectrum is usually discrete.) In the subsections to follow we will rederive the semiclassical result given in the previous section and then calculate the reflection and transmission coefficients using the results of section (II A 3). We then look for a correspondence between the reflection and transmission coefficients with the semiclassical results.

1. Semiclassical analysis of the effective Schrodinger equation

The semiclassical analysis follows closely the method adopted in section (IV A). The action functional $\mathcal{A}$ for a classical particle moving in a potential $-g/x^2$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial\mathcal{A}}{\partial t} + \left(\frac{\partial\mathcal{A}}{\partial x}\right)^2 - \frac{g}{x^2} = 0 \tag{93}$$

The solution can be immediately written down as,

$$\mathcal{A} = -\widetilde{E}t \pm \int^x \frac{dx}{x}\sqrt{\widetilde{E}x^2 + g} \tag{94}$$

Eqn (94) has an integral which is divergent if the action is computed for points lying on the opposite sides of the horizon $x = 0$. Since this has a similar form to Eqn (71), the prescription used in evaluating $S[\text{emission}]$ and $S[\text{absorption}]$ can be similarly used to evaluate $\mathcal{A}[\text{emission}]$ and $\mathcal{A}[\text{absorption}]$. The results are

$$\mathcal{A}[\text{emission}] = i\pi\sqrt{g} + (\text{real part})$$
$$\mathcal{A}[\text{absorption}] = -i\pi\sqrt{g} + (\text{real part}). \tag{95}$$

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Constructing the semiclassical propagator as before and taking the modulus square to obtain the probabilities for outgoing and ingoing particles, we get

$$P[\text{emission}] = \exp\left[-4\pi\sqrt{\frac{1}{4} + \frac{E^2}{\hbar^2 R^2}}\right] P[\text{absorption}]. \tag{96}$$

Note that the result is independent of $\widetilde{E}$. Therefore, the above result holds even in the case $\widetilde{E} = 0$. The formula above does not represent a thermal system since the energy E appears in the squareroot. For large energies such that $(E^2/\hbar^2 R^2) \gg (1/4)$, the above equation can be written in the form that suggests a thermal system

$$P[\text{outgoing}] = \exp\left[-\frac{4\pi E}{\hbar R}\right] P[\text{ingoing}]. \tag{97}$$

The temperature β^{-1} for the system is the same as that in Eqn (82) and one recovers the usual result. For the case of the Schwarzschild black hole, the condition $(E^2/\hbar^2 R^2) \gg (1/4)$ reduces to $(E/\hbar) \gg 1/4M$. In proper units, it can be written as

$$E \gg \frac{\hbar c^3}{4GM} = M_P c^2 \frac{M_P}{4M} \tag{98}$$

where $M_P = \sqrt{\hbar c/G}$ is the Planck mass. Since the energy E must be much smaller than the Planck energy $M_P c^2$ in the semiclassical limit, the condition above reduces to $M \gg M_P$ i.e. the mass of the black hole must be far greater than the Planck mass for the spectrum to be thermal. This differs significantly from the result given in Eqn (80) which does not put any condition on the size of the black hole for the spectrum to be thermal. To verify that the semi-classical analysis is valid, one must compute the correction terms and check that these have a singularity of the same order as possessed by $\mathcal{A}$. To do this, consider the effective Schrodinger equation (92) with factors of $\hbar$ put in.

$$-\hbar^2 \frac{d^2 Q(x)}{dx^2} - \frac{g}{x^2} Q(x) = \widetilde{E} Q(x) \tag{99}$$

Putting $Q(x) = \exp(iA(x)/\hbar)$, and substituting into Eqn (99),

$$-i\hbar \frac{d^2 A(x)}{dx^2} + \left(\frac{dA(x)}{dx}\right)^2 = \widetilde{E} + \frac{g}{x^2}. \tag{100}$$

Expanding A in powers of $\hbar/i$, we get,

$$A = \mathcal{A} + \frac{\hbar}{i} A_1 + \left(\frac{\hbar}{i}\right)^2 A_2 + \dots \tag{101}$$

Substituting into Eqn (100) and proceeding as usual, we find that $\mathcal{A}$ is given by Eqn (94). The next term A_1 is given by

$$A_1 = g \int \frac{dx}{x} \frac{1}{\widetilde{E} x^2 + g}. \tag{102}$$

The relation for A_1 also has a singularity at the origin of the same order as $\mathcal{A}$. Explicit calculation of the subsequent terms in the expansion of A reveals that all these terms have a singularity of the same order as that of $\mathcal{A}$ and therefore their net contribution to the kernel is either as phase factors or as the exponential of finite terms multiplied by powers of $\hbar$. Therefore, we conclude as before that the semiclassical ansatz is valid.

2. Tunneling Interpretation

Instead of the purely semiclassical derivation given above, we attempt to see if the relation between the reflection and transmission coefficients for the effective potential calculated above could be interpreted to give a thermal spectrum. Using the results in section (II A 3) and in particular Eqn. (37) after setting $2m = \hbar = 1$, we obtain,

$$R = T e^{-2\pi\sqrt{g}} = T \exp\left[-\frac{2\pi E}{\hbar R}\right] \tag{103}$$

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for "large" energies satisfying the inequality Eqn. (98). We may interpret the transmission coefficient as the probability of emission of particles while the reflection coefficient as that for absorption but we obtain a temperature which is twice the temperature obtained using the semiclassical analysis. It is also clear why this is so. The transmission and absorption of particles to and from the horizon is clearly not symmetric. In the semiclassical analysis, this asymmetry was put in by hand in the form of the prescription used to evaluate the semiclassical kernel while in the tunneling analysis, the potential near the horizon is completely symmetric and there is no apriori reason for the transmission coefficient to be different on either side of the horizon. The tunnelling interpretation, in the case of the electric field, works in both gauges because of the structural similarity of the effective Schrodinger equation in both cases. Further, in the time dependent gauge, boundary conditions play an important role in determining particle creation. The boundary condition that, at $t = -\infty$, the quantum field is initially in a vacuum state, introduces an asymmetry into the problem which gives rise to particle production in the infinite future as $t \to \infty$. But, in the space dependent gauge, such a time asymmetry in the boundary conditions cannot be implemented since the problem is time independent. But, since the structure of the effective schrodinger equation to be solved is the same as that in the time dependent gauge, suitable ratios of the transmission and reflection coefficients, which are symmetric in both space directions, can be identified with the Bogolibov coefficients with the subsequent recovery of particle production.

In the Schwarzchild spacetime, the analogue of the "time dependent gauge" is provided by the study of a collapsing shell of matter [2] which, close to the black hole horizon, gives rise to production of particles which have a thermal distribution in energy. In this case, the boundary condition considered at the collapsing shell causes an exponential redshift of incoming modes. The Bogolibov coefficients can be calculated in a straight forward manner and $|\beta_\lambda|^2$ is Planckian in form and particle production is established in the same manner as was done in the electric field case. The analogue of the "space-dependent gauge" is provided by the system where an eternal black hole is in equilibrium with a bath of thermal radiation. The transmission and reflection coefficients are easily determined in this case. But, if we attempt to identify the Bogoliubov coefficients using ratios of these coefficients, we obtain the result in Eqn. (103) where the tunnelling temperature is twice Hawking's result. On the other hand, if we use the correspondence given in section (III) for the identification we obtain results similar to those obtained for the electric field and this does not yield a thermal spectrum for the emitted particles. A possible qualitative explanation for the factor 2 in the tunnelling temperature is that, since there are two disjoint Schwarzschild sectors in the full Kruskal manifold, the energy E in Eqn. (103) should be multiplied by a factor 2 resulting in the recovery of Hawking's result. Therefore, the radiation energy seen at infinity is the result of tunnelling in both sectors. Such an explanation holds for all the spacetimes considered in this paper with a horizon, namely, the Rindler and the de-Sitter where the full manifold consists of two mutually disjoint but identical spacetimes. But a more quantitative and satisfactory tunnelling interpretation in black hole like spacetimes is needed.

V. COMPLEX PATHS INTERPRETATION OF PARTICLE PRODUCTION IN ELECTRIC FIELD

In section (III), we noted how particle production can be calculated using the tunnelling interpretation. This interpretation gives the same result in both the space as well as the time dependent gauges. The spectrum of particles produced by the electric field is not thermal in contrast to the spectrum seen by a Rindler observer. We use the formal tunnelling method to show, in a heuristic manner, how this particle production can be obtained by tunnelling between the two sectors of the Rindler spacetime.

The imaginary part of the effective lagrangian $\mathrm{ImL_{eff}}$, which is related to the probability of the system to remain in the vacuum state for all time is given by [3]

$$\mathrm{ImL_{eff}} = \sum_{n=1}^{\infty} \frac{1}{2} \frac{(qE_0)^2}{(2\pi)^3} \frac{(-1)^{n+1}}{n^2} \exp\left(-\frac{\pi m^2}{qE_0}n\right) \tag{104}$$

where m is the mass, q is the charge and E_0 is the magnitude of the electric field. We will derive the above expression for $\mathrm{ImL_{eff}}$ using the general arguments given in section (II A).

Consider the Hamilton-Jacobi equation for the motion of a particle in an electromagnetic field in $(1+1)$ dimensions with a proper time co-ordinate s.

$$\frac{1}{2}\left(\frac{\partial F}{\partial t} + qA^t\right)^2 - \frac{1}{2}\left(\frac{\partial F}{\partial x} - qA^x\right)^2 - \frac{1}{2}m^2 + \frac{\partial F}{\partial s} = 0 \tag{105}$$

where F is the action and $A^\mu = (A^t, A^x, 0, 0)$ is the four vector potential. We have neglected the dependence of F on the y and z co-ordinates. This will be justified later on. In a time-dependent gauge, given by Eqn. (42), the action F can be easily solved for by using the ansatz $F = -Es + p_x x + f(t)$ to give,

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$$F(t_1, x_1; t_0, x_0; s) = -Es + p_x(x_1 - x_0) \pm \int_{t_0}^{t_1} dt \sqrt{(p_x + qE_0 t)^2 + (m^2 + 2E)} \tag{106}$$

where p_x is the momentum of the particle in the x direction and E is the energy of the particle corresponding to the proper time co-ordinate s. The trajectory of the particle in the (t, x) plane is the usual hyperbolic trajectory given by,

$$(t - t_i)^2 - (x - x_i)^2 = -k_0^2 \tag{107}$$

where t_i, x_i and k_0 are suitable constants. For any fixed position x, there are thus two disjoint trajectories corresponding to motion in the two Rindler wedges.

Let us consider the tunnelling of a particle from one Rindler trajectory to the other *and back* in the imaginary time co-ordinate t. This means that the particle comes back to the same spacetime point as it started from. Since the process is a tunnelling process, the proper time taken is zero. Therefore, choosing the positive sign in Eqn. (106) (this choice give a tunnelling probability that is exponentially damped), we have,

$$\begin{aligned}
F(t_0, x_0; t_0, x_0; 0) &= \oint dt \sqrt{(p_x + qE_0 t)^2 + (m^2 + 2E)} = \frac{m^2 + 2E}{qE_0} \oint du \sqrt{1 + u^2} \\
&= i\frac{m^2 + 2E}{qE_0} \oint d\tau \sqrt{1 - \tau^2} = i\frac{m^2 + 2E}{qE_0} \int_0^{2\pi} d\theta \cos^2(\theta) \\
&= \frac{i\pi(m^2 + 2E)}{qE_0}
\end{aligned} \tag{108}$$

where we have made the following changes of variable $(p_x + qE_0 t) = u = i\tau$ and $\tau = \sin(\theta)$. Taking the limit $E \to 0$, the expression for $\exp(iF)$ from the above equation is seen to be exactly the same as the exponential term in Eqn. (104) for $n = 1$. The same argument can be repeated for the particle tunnelling n times to and fro to give

$$F_n(t_0, x_0; t_0, x_0; 0) = i\frac{m^2 + 2E}{qE_0} \int_0^{2n\pi} d\theta \cos^2(\theta) = \frac{i\pi(m^2 + 2E)}{qE_0} n. \tag{109}$$

Again, in the limit $E \to 0$, $\exp(iF_n)$ is seen to match with the exponential part of the nth term in Eqn. (104). Therefore, the imaginary part of the total effective lagrangian can be written down immediately as

$$\mathrm{ImL}_{\mathrm{eff}} = \sum_{n=1}^{\infty} (\text{prefactor}) \exp\left(-\frac{\pi m^2}{qE_0} n\right) \tag{110}$$

where the prefactor can only be calculated using the exact kernel. The dependence of the prefactor on n and the phase factor $(-1)^{n+1}$ in Eqn. (104) can be deduced using the following arguments.

The formal expression of the path integral kernel for the above electric field problem in the time dependent gauge is given by [3]

$$K(a, b; s) = \langle a | e^{ish} | b \rangle \tag{111}$$

where $K(a, b; s)$ is the kernel for the particle to propagate between the spacetime points $a = (x^0, \mathbf{x})$ and $b = (y^0, \mathbf{y})$ in a proper time s and h is the hamiltonian given by

$$h = \frac{1}{2}(i\partial_i - qA_i)(i\partial^i - qA^i) - \frac{1}{2}m^2 \tag{112}$$

where A^i is the four vector potential given in Eqn. (42) and q and m are the charge and mass of the particle respectively. Going over to momentum co-ordinates and considering the coincidence limit $\mathbf{x} = \mathbf{y}$, the kernel can be written in the form,

$$K(x^0, y^0; \mathbf{x}, \mathbf{x}; s) = -\frac{i}{2(2\pi)^2} \int_{-\infty}^{\infty} \frac{dp_x}{s} \mathcal{G}(x^0, y^0; s) \tag{113}$$

where $\mathcal{G}(x^0, y^0; s)$ is given by

$$\mathcal{G}(x^0, y^0; s) = \langle x^0 | e^{isH} | y^0 \rangle \tag{114}$$

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and H is the hamiltonian

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + (p_x + qE_0 t)^2 + m^2\right) = -\frac{1}{2}\left(\frac{\partial^2}{\partial \rho^2} + q^2 E_0^2 \rho^2 + m^2\right) \tag{115}$$

with $\rho = (t + p_x/qE_0)$. In the expression for the kernel in momentum co-ordinates, we have integrated over the transverse momentum variables p_y and p_z. The above hamiltonian is that of an inverted harmonic oscillator. Since all reference to p_y and p_z have disappeared in H, the dependence of F on the y and z co-ordinates was neglected when writing down the expression for the Hamilton-Jacobi equation in Eqn. (105). The expression for the effective lagrangian is then given by

$$L_{\text{eff}} = -i\int_0^\infty \frac{ds}{s} K(x^0, x^0; \mathbf{x}, \mathbf{x}; s) = -\frac{1}{2(2\pi)^2}\int_0^\infty \frac{ds}{s^2}\int_{-\infty}^\infty dp_x \mathcal{G}(x^0, x^0; s) \tag{116}$$

We would like to evaluate the propagator $\mathcal{G}(x^0, x^0; s)$ for a tunnelling situation where the particle tunnels from the point x^0 and back in loops. Since the path integral is not well defined for closed loops, it will have to be evaluated in some approximate limiting procedure which is outlined below.

Since the tunnelling potential is that of an inverted oscillator, we can use all the results of section (II A 2). The semi-classical wave functions are given in Eqns. (20, 21). We would like to first account for the factor $(-1)^n$ that arises when a particle tunnels from one side of the barrier to the other and back. Consider an incident wave to the right of the barrier and impinging on it. Using the method of complex paths, we rotate this wave in the lower complex plane (this is the only route possible for the same reason as that given when rotating a right moving travelling wave in the upper complex plane) to obtain a wave again incident on the barrier with a energy independent phase factor $\exp(i\pi/2)$ being picked up (other factors dependent on the energy are also picked up but these are not important here). Since this wave is moving in the wrong direction, we assume that the particle that has tunnelled through has the same amplitude as the rotated wave but is moving away from the barrier. This just involves changing the sign of the argument of the exponential in the expression for the rotated wave. Rotating this left moving wave again in the upper complex plane now, the final wave obtained is a right moving wave with another extra phase factor of $\exp(i\pi/2)$ being picked up. The total phase change with respect to the incident wave is thus $\exp(i\pi)$. Since this phase factor is independent of the energy, the propagator for the tunnelling process too, after one such rotation, will pick up a phase of $\exp(i\pi)$. Similarly, for n rotations, n taking the values $1, 2, 3, \ldots$, the phase acquired will be $\exp(in\pi) = (-1)^n$.

Therefore, the propagator $\mathcal{G}$ for n loops, $\mathcal{G}_n(x^0, x^0; s)$, can thus be written as

$$\mathcal{G}_n(x^0, x^0; s) = N(p_x, m, E)e^{in\pi}e^{iF_n(x^0, x^0; s)} \tag{117}$$

where $F_n(x^0, x^0; s)$ is the classical action for the tunnelling process and N is the prefactor that arises after evaluating the "sum over paths". This prefactor is not expected to depend on the proper time s since the tunnelling process takes place instantaneously or on the number of rotations n. So the only quantities it may depend on are p_x, m and E. Now the expression for the classical action for n loops is given in Eqn. (109) with E set to zero. Substituting this into the expression for the propagator in Eqn. (117) and evaluating $L_{\text{eff}}(n)$ for n loops from the relation in Eqn. (116) we obtain infinity as the answer which is meaningless. Therefore, we adopt the following limiting procedure in order to obtain a finite and meaningful answer. Consider the action for the tunnelling process.

$$F = -Es + p_x x \pm \int dt \sqrt{(p_x + qE_0 t)^2 + (m^2 + 2E)} \tag{118}$$

Choosing the positive sign and setting $p_x + qE_0 t = i\sqrt{m^2 + 2E}\sin(\theta)$, we obtain,

$$F = -Es + p_x x \pm i\frac{m^2 + 2E}{2qE_0}\int d\theta(1 + \cos(2\theta)) \tag{119}$$

For *closed* paths only, with θ taking the values from 0 to $2n\pi$, the above action can be written as,

$$F_n = -Es \pm i\frac{m^2 + 2E}{2qE_0}\theta \tag{120}$$

We have thrown away the $p_x x$ term while retaining the $-Es$ term since the dependence on the x co-ordinate is really irrelevant. Defining a new variable $\bar{\theta} = i\theta/2qE_0$ and rescaling $s = \alpha s'$, the above action becomes

$$F_n = -E\alpha s' + (m^2 + 2E)\bar{\theta}. \tag{121}$$

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Choosing α appropriately, the above action can be cast into a form that matches the action for a fictitious free particle in (1+1) dimensions with "energy" αE and "momentum" (m^2+2E) satisfying the classical energy-momentum relation,

$$\alpha E = \frac{1}{2}(m^2 + 2E)^2 \tag{122}$$

where the particle's "mass" is set to unity for convenience. The above equation determines the quantity α. Therefore, F_n can also be written in the form

$$F_n(\bar{\theta}_2, \bar{\theta}_1; s) = \frac{(\bar{\theta}_2 - \bar{\theta}_1)^2}{2s'} = \frac{\alpha(\bar{\theta}_2 - \bar{\theta}_1)^2}{2s} \tag{123}$$

where $\bar{\theta}_1$ and $\bar{\theta}_2$ are the initial and final states of the free particle with s' being the proper time taken. (Note that $(\bar{\theta}_2 - \bar{\theta}_1) = 2in\pi/2qE_0$.) Substituting this into the expression for $\mathcal{G}_n(x^0, x^0; s)$ in Eqn. (117) and evaluating only the integral over s in the expression for the effective lagrangian in Eqn. (116) *without* taking the limits, we obtain

$$\int \frac{ds}{s^2} \mathcal{G}_n(x^0, x^0; s) = N(p_x, m, E) e^{in\pi} \int \frac{ds}{s^2} \exp\left(\frac{i\alpha(\bar{\theta}_2 - \bar{\theta}_1)^2}{2s}\right)$$

$$= -N(p_x, m, E) e^{in\pi} \frac{2}{i\alpha(\bar{\theta}_2 - \bar{\theta}_1)^2} \exp\left(iF_n(\bar{\theta}_2, \bar{\theta}_1; s)\right). \tag{124}$$

Notice that the prefactor to the exponential has no dependence on the proper time s. Now, we use the form for the action given by Eqn. (120) and taking the limits for s from 0 to ∞, the effective lagrangian for n loops, $L_{\text{eff}}(n)$ can be written in the form,

$$L_{\text{eff}}(n) = \frac{i}{2} \frac{(qE_0)^2}{(2\pi)^3} \frac{(-1)^{n+1}}{n^2} \exp\left(-\frac{\pi m^2}{qE_0} n\right) \frac{8}{\alpha} \exp\left(-\frac{2\pi E}{qE_0} n\right) \int_{-\infty}^{\infty} \frac{dp_x}{2\pi} N(p_x, m, E) \tag{125}$$

where we have set $(\bar{\theta}_2 - \bar{\theta}_1) = 2in\pi/2qE_0$. Taking the limit of $E \to 0$ and using the expression for α in Eqn. (122), we find that N must satisfy the relation

$$\lim_{E \to 0} \frac{16E}{(m^2 + 2E)^2} \int_{-\infty}^{\infty} \frac{dp_x}{2\pi} N(p_x, m, E) = 1 \tag{126}$$

so that the imaginary part of the effective lagrangian for n loops, $L_{\text{eff}}(n)$, matches the nth term in Eqn. (104). Therefore, in this manner, the contributions to the imaginary part of the effective lagrangian for the uniform electric field can be thought of as arising from the tunnelling of particles between the two Rindler sectors.

VI. CONCLUSIONS

In conclusion, we see that the tunnelling picture is not a valid picture in general. In the case of the electric field, it can successfully be applied to both the time dependent and time independent gauges but such a picture is valid only to a limited extent. When applied naively to black hole like spacetimes it gives a temperature that is twice the expected value. However, by taking into account the presence of the two disjoint Schwarzchild sectors in the full Kruskal manifold this discrepancy can be corrected. The semi-classical prescription given in the paper takes into account the asymmetry between the two sides of the horizon and gives the correct result without requiring the Kruskal extension. An interesting aspect in the reduction to the effective Schrodinger problem is that the semi-classical analysis after the reduction does not give a thermal spectrum for all energies. Only when the mass of the black hole is much greater than the Planck mass does the spectrum reduce to a thermal form.

But the tunnelling picture does seem to give a nice interpretation of particle production in an electric field as arising due to tunnelling between the two disjoint sectors of the Rindler spacetime. Though we have only given a heuristic argument in this paper, we will explore this issue further in a future publication.

ACKNOWLEDGEMENTS

APPENDIX: GENERALIZATION TO (3+1)–DIMENSIONS

The generalization to (3+1) dimensions is straightforward. In this section, the semi-classical analysis given in section (IV A) will be briefly outlined. Further, it will be shown that the effective Schrodinger equation in (3+1) dimensions is the same as Eqn (91).

Consider the metrics (59) and (60). We will work with (59) which is in spherical polar co-ordinates. The results obtained are extendable to (60) in a straightforward manner.

The Klein-Gordon equation, written using the metric (59), is

$$
\frac{1}{B(r)}\frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B(r)\frac{\partial \Phi}{\partial r}\right) - \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Phi}{\theta}\right)
$$
$$
- \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{m_0^2}{\hbar^2}\Phi \tag{127}
$$

Since the problem is a spherically symmetric one, one can put $\Phi = \Psi(r,t)Y_l^m(\theta,\phi)$ to obtain,

$$
\frac{1}{B(r)}\frac{\partial^2 \Psi}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B(r)\frac{\partial \Psi}{\partial r}\right) + \left(\frac{l(l+1)}{r^2} + \frac{m_0^2}{\hbar^2}\right)\Psi = 0 \tag{128}
$$

Making the ansatz $\Psi = \exp((i/\hbar)S(r,t))$ and substituting into the above equation, we obtain,

$$
\left[\frac{1}{B(r)}\left(\frac{\partial S}{\partial t}\right)^2 - B(r)\left(\frac{\partial S}{\partial r}\right)^2 - m_0^2 - \frac{l(l+1)\hbar^2}{r^2}\right]
$$
$$
+\frac{\hbar}{i}\left[\frac{1}{B(r)}\frac{\partial^2 S}{\partial t^2} - B(r)\frac{\partial^2 S}{\partial r^2} - \frac{1}{r^2}\frac{d(r^2 B)}{dr}\frac{\partial S}{\partial r}\right] = 0 \tag{129}
$$

Expanding S in a power series as in Eqn (69), we obtain, to the zeroth order in $\hbar/i$,

$$
S_0 = -Et \pm \int^r \frac{dr}{B(r)}\sqrt{E^2 - B(r)(m_0^2 + L^2/r^2)} \tag{130}
$$

where $L^2 = l(l+1)\hbar^2$ is the angular momentum. It is easy to see from the above equation that near the horizon, the presence of the L^2 term can be neglected since it is multiplied by $B(r)$. Therefore, the semiclassical result of section (IV A) follows even in the case of (3+1)–dimensions. The semi-classical ansatz is valid in this case as can be seen by calculating explictly the higher order terms in the expansion for S. All these terms have a singularity at the horizon of the same order as that of S_0 and they contribute to the semiclassical propagator either as phase factors or as terms multiplied by powers of $\hbar$ which are entirely negligible. Expanding the Klien-Gordon equation for Φ using the metric (60) gives analogous results and will not be explictly given.

Now, consider the reduction of Eqn (127) to an effective Schrodinger equation. Setting

$$
\Psi = \exp(-iEt/\hbar)Y_l^m(\theta,\phi)\Psi(r) \tag{131}
$$

and substituting into Eqn (127), we obtain,

$$
B(r)\frac{d^2\Psi}{dr^2} + \frac{1}{r^2}\frac{d(r^2 B)}{dr}\frac{d\Psi}{dr} + \left(\frac{E^2}{\hbar^2 B(r)} - \frac{m^2}{\hbar^2} - \frac{L^2}{\hbar^2 r^2}\right)\Psi = 0 \tag{132}
$$

Making the substitution

$$
\Psi = \frac{1}{\sqrt{r^2 B(r)}}Q(r) \tag{133}
$$

we get the result,

$$
-\frac{d^2 Q}{dr^2} - \left[\frac{1}{B^2}\left(\frac{(B')^2}{4} + \frac{E^2}{\hbar^2}\right) - \frac{1}{B}\left(\frac{B''}{2} + \frac{B'}{r} + \frac{m_0^2}{\hbar^2} + \frac{L^2}{\hbar^2 r^2}\right)\right]Q = 0 \tag{134}
$$

where B' and B'' are the first and second derivatives of $B(r)$ respectively. Near the horizon $r = r_0$, using the expansion for $B(r)$ given in Eqn (64), it is easy to see that the $1/B^2$ term in the above equation dominates over the $1/B$ term

which can therefore be neglected. The resulting Schrodinger equation is the same as in Eqn (91). Therefore, even in (3+1) dimensions, the semiclassical and quantum mechanical results of section (IV B) are the same. It can be easily proved that the effective Schrodinger equation for the cartesian metric (60) gives the same result.

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